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1999 J. Phys. A: Math. Gen. 32 2427

(http://iopscience.iop.org/0305-4470/32/12/015)

J. Phys. A: Math. Gen. **32** (1999) 2427–2445. Printed in the UK PII: S0305-4470(99)00109-2

Geometric factors in the Bohr–Rosenfeld analysis of the measurability of the electromagnetic field

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Received 9 December 1998

Abstract. The geometric factors in the field commutators and spring constants of the measurement devices in the famous analysis of the measurability of the electromagnetic field by Bohr and Rosenfeld are calculated using a Fourier–Bessel method for the evaluation of folding integrals, which enables one to obtain the general geometric factors as a Fourier–Bessel series. When the space regions over which the factors are defined are spherical, the Fourier–Bessel series terms are given by elementary functions, and using the standard Fourier-integral method of calculating folding integrals, the geometric factors can be evaluated in terms of manageable closed-form expressions.

1. Introduction

The fundamental importance of the famous paper of Bohr and Rosenfeld [1] on the measurability of the electromagnetic field is acknowledged by most physicists but, curiously, the paper seems to have been read by only a few at the time of its appearance in the early 1930s†, and undoubtedly by still less in the more recent times‡. The Bohr–Rosenfeld (BR) paper arose in a response to Landau and Peierls [3], who argued that, in principle, the electromagnetic field is not measurable in a domain where quantum and relativistic effects are important. In their paper, BR have refuted this claim by showing that in quantum electrodynamics, just as in nonrelativistic quantum mechanics, there is a complete harmony between the theoretical formalism and the physical possibilities of measurement. The essential aspects of the BR analysis that enabled them to reach this conclusion are the realization that only field quantities averaged over finite space-time regions are physically meaningful, and the employment of essentially classical test bodies of finite size that can have arbitrarily large charge and mass, which removed immediately the limit on the field measurability due to the radiation-reaction force on the point test charges employed by Landau and Peierls. BR have shown that the electromagnetic-field effects of such finite-size test bodies can be minimized to exactly the extent demanded by the formalism's commutation relations by using classical spring and lever mechanisms connecting the test bodies to the frame of reference and to each other, together

† A Pais says in his book on N Bohr [2]: 'From decades of involvement with quantum field theory I can testify that nevertheless it has been read by very very few of the *aficionados*. The main reason is, I think, that even by Bohr's standards this paper is very difficult to penetrate. It takes inordinate care and patience to follow Bohr's often quite complex gyrations with test bodies. . . As a friend of Bohr and mine once said to me: 'It is a very good paper that one does not have to read. You just have to know it exists.' Nevertheless men like Pauli and Heitler did read it with great care.'

‡ A contributing factor to that must have been that the paper had not been available in an English translation for a long time.

0305-4470/99/122427+19$19.50

with the deployment of neutralizing bodies that occupy the same space regions as the test bodies but which are charged oppositely and remain attached rigidly to the reference frame during the duration of a measurement. The harmony between the possibilities of definition afforded by the quantum-electrodynamic formalism and the possibilities of measurement could be attained only by the masterful exploitation by BR of all the opportunities offered to measurement by classical physics while at the same time paying due regard to the limitations imposed on the latter by quantum mechanics†.

Several illuminating commentaries on the BR analysis have been written [2, 5–7], some of them by authors who were close to the original controversy. While the BR field measurement philosophy has not been accepted unreservedly by all the writers [6], the technical correctness of the BR analysis has not been disputed. Very recently, however, an analysis [8] of the BR procedure for the measurement of a single space-time-averaged component of the electromagnetic field has drawn a conclusion that no compensating spring mechanism is needed in order to measure the averaged field component to arbitrary accuracy when no neutralizing body is employed. This work is commented on critically elsewhere [9], using the calculational methods developed in this paper.

The field commutation relations that BR use as the starting point of their analysis, and the spring constants of the mechanisms employed in their measurement procedures, are defined in terms of geometric factors that are averages over two finite four-dimensional space-time regions. As this amounts formally to an eight-dimensional integration, the calculation of the value of a field commutator for finite space-time regions, or of a BR spring constant, is not simple, even though the dimensionality of the integration is reduced, albeit not in a straightforward manner, by the presence of a delta function in the integrand. To the present author's knowledge, no calculations of these quantities have been reported in the literature yet, apart from those for coinciding spherical space-time regions in the comment [9] on [8]. Clearly, a well-controlled algorithm for the evaluation of the field commutators for finite space-time regions and the BR spring constants is desirable—not least because these quantities amount essentially to the field effects of extended charged bodies, a quantitative assessment of which may be needed in special experimental situations‡.

In this paper, the Fourier–Bessel method developed for an efficient and accurate evaluation of multiple folding integrals [10] is adapted for the calculation of general BR geometric factors. However, the standard Fourier-integral method for calculating folding integrals will turn out to be more advantageous in the special case of spherical space regions, as it will enable us to evaluate the BR geometric factors with spherical space regions in closed form.

In section 2 of this paper, the BR field commutators are introduced, and the Fourier–Bessel method for calculating the general geometric factors in terms of which the field commutators and the BR spring constants are defined is developed. In section 3, geometric factors with spherical space regions are considered; these are first calculated using Fourier–Bessel expansions, and then evaluated in closed form. In the last section, some calculational results are presented and discussed, and concluding remarks are made.

† In the course of their analysis, BR had to examine the problem of measurement of the basic mechanical quantities of position, momentum and energy in more detail than had been done in the previous writings of Bohr and Heisenberg, and established the possibility of repeatable momentum and energy measurements that may be of arbitrarily short duration, which were rediscovered some 30 years later by Aharonov and Bohm [4].

‡ For example, direct detection of gravitational waves would require repeated measurements of very high accuracy on a single object in a regime where quantum effects are important. In these so-called 'quantum nondemolition' measurements, experimental precision is pushed to the limits set by the principles of quantum mechanics and quantum electrodynamics; under such or similar circumstances, the measurement procedures and results of the BR analysis may well be of practical relevance [11].

2. Fourier–Bessel expansions of the BR geometric factors

The starting point in the BR analysis is the set of electromagnetic-field commutation relations for the operators of field averages over finite space-time regions instead of for those of the field values at space-time points:

$$[\bar{\mathcal{E}}_x^{(\mathrm{I})}, \bar{\mathcal{E}}_x^{(\mathrm{II})}] = [\bar{\mathcal{H}}_x^{(\mathrm{I})}, \bar{\mathcal{H}}_x^{(\mathrm{II})}] = \mathrm{i}\hbar(\bar{A}_{xx}^{(\mathrm{I,II})} - \bar{A}_{xx}^{(\mathrm{II,I})}) \tag{1}$$

$$[\bar{\mathcal{E}}_x^{(\mathrm{I})}, \bar{\mathcal{E}}_y^{(\mathrm{II})}] = [\bar{\mathcal{H}}_x^{(\mathrm{I})}, \bar{\mathcal{H}}_y^{(\mathrm{II})}] = \mathrm{i}\hbar(\bar{A}_{xy}^{(\mathrm{I,II})} - \bar{A}_{xy}^{(\mathrm{II,I})}) \tag{2}$$

$$[\bar{\mathcal{E}}_x^{(\mathrm{I})}, \bar{\mathcal{H}}_x^{(\mathrm{II})}] = 0 \tag{3}$$

$$[\bar{\mathcal{E}}_x^{(\mathrm{I})}, \bar{\mathcal{H}}_y^{(\mathrm{II})}] = -[\bar{\mathcal{H}}_x^{(\mathrm{I})}, \bar{\mathcal{E}}_y^{(\mathrm{II})}] = \mathrm{i}\hbar(\bar{B}_{xy}^{(\mathrm{I,II})} - \bar{B}_{xy}^{(\mathrm{II,I})}). \tag{4}$$

Here, $\bar{\mathcal{E}}_x^{(\mathrm{I})}$, $\bar{\mathcal{H}}_x^{(\mathrm{I})}$, etc are the electric and magnetic field components averaged over a space-time region I of volume V_I and duration T_I, as, for example,

$$\bar{\mathcal{E}}_x^{(\mathrm{I})} = \frac{1}{V_\mathrm{I} T_\mathrm{I}} \int_{T_\mathrm{I}} \mathrm{d}t_1 \int_{V_\mathrm{I}} \mathrm{d}v_1\, \mathcal{E}_x(x_1, y_1, z_1, t_1) \tag{5}$$

and the right-hand-side quantities are purely *geometric factors* defined in terms of double averages over space-time regions I and II:

$$\bar{A}_{xx}^{(\mathrm{I,II})} = -\frac{1}{\mathcal{V}_{\mathrm{I,II}}} \int_{T_\mathrm{I}} \mathrm{d}t_1 \int_{T_\mathrm{II}} \mathrm{d}t_2 \int_{V_\mathrm{I}} \mathrm{d}v_1 \int_{V_\mathrm{II}} \mathrm{d}v_2 \left(\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{1}{c^2} \frac{\partial^2}{\partial t_1 \partial t_2} \right) \left[\frac{1}{r} \delta \left(t_2 - t_1 - \frac{r}{c} \right) \right] \tag{6}$$

$$\bar{A}_{xy}^{(\mathrm{I,II})} = -\frac{1}{\mathcal{V}_{\mathrm{I,II}}} \int_{T_\mathrm{I}} \mathrm{d}t_1 \int_{T_\mathrm{II}} \mathrm{d}t_2 \int_{V_\mathrm{I}} \mathrm{d}v_1 \int_{V_\mathrm{II}} \mathrm{d}v_2 \frac{\partial^2}{\partial x_1 \partial y_2} \left[\frac{1}{r} \delta \left(t_2 - t_1 - \frac{r}{c} \right) \right] \tag{7}$$

$$\bar{B}_{xy}^{(\mathrm{I,II})} = -\frac{1}{\mathcal{V}_{\mathrm{I,II}}} \int_{T_\mathrm{I}} \mathrm{d}t_1 \int_{T_\mathrm{II}} \mathrm{d}t_2 \int_{V_\mathrm{I}} \mathrm{d}v_1 \int_{V_\mathrm{II}} \mathrm{d}v_2 \frac{1}{c} \frac{\partial^2}{\partial t_1 \partial z_2} \left[\frac{1}{r} \delta \left(t_2 - t_1 - \frac{r}{c} \right) \right] \tag{8}$$

where $\mathcal{V}_{\mathrm{I,II}} = V_\mathrm{I} V_\mathrm{II} T_\mathrm{I} T_\mathrm{II}$ and r is the distance between a space point in the region I and a space point in the region II. The remaining commutation relations are obtained from (1)–(4) and (6)–(8) by cyclic permutations.

A BR geometric factor, say $\bar{C}_U^{(\mathrm{I,II})}$, can be written as

$$\bar{C}_U^{(\mathrm{I,II})} = \frac{1}{\Delta t_1 \Delta t_2} \int_0^{\Delta t_1} \mathrm{d}t_1 \int_T^{T+\Delta t_2} \mathrm{d}t_2 \int \rho_1(\boldsymbol{r}_1)\, \mathrm{d}\boldsymbol{r}_1 \int \rho_2(\boldsymbol{r}_2)\, \mathrm{d}\boldsymbol{r}_2\, U(t, \boldsymbol{r}). \tag{9}$$

Here, the time intervals associated with the space-time regions I and II are specified, without loss of generality, as $(0, \Delta t_1)$ and $(T, T + \Delta t_2)$, respectively†, while the space regions are given by the means of uniform density distributions $\rho_1(\boldsymbol{r}_1)$ and $\rho_2(\boldsymbol{r}_2)$ that vanish outside the space regions I and II, respectively, and are each normalized to unit volume; the coordinates $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ now refer to origins located conveniently inside the regions I and II so that the displacement $\boldsymbol{r}$ of a space point of region II from a space point of region I is given as $\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{r}_2 - \boldsymbol{r}_1$, with $\boldsymbol{R}$ the displacement of the origin of region II from that of region I. The function $U(t, \boldsymbol{r})$ in equation (9) is the integrand of the multiple integral defining the geometric factor; for the geometric factors (6)–(8), it takes the following forms, respectively:

$$U_{A_{xx}}(t, \boldsymbol{r}) = -\left(\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{\partial t_1 \partial t_2} \right) \frac{\delta(t - r)}{r} \tag{10}$$

$$U_{A_{xy}}(t, \boldsymbol{r}) = -\frac{\partial^2}{\partial x_1 \partial y_2} \frac{\delta(t - r)}{r} \tag{11}$$

† BR use the symbols T_I and T_II for these time intervals, and the symbol Δt for the durations of the momentum measurements at the beginning and end of a field-measurement time interval.

$$U_{B_{xy}}(t, \boldsymbol{r}) = -\frac{\partial^2}{\partial t_1 \partial z_2} \frac{\delta(t-r)}{r} \tag{12}$$

where units such that the speed of light $c = 1$ are now used, and $t = t_2 - t_1$. The BR geometric factor (9) is cast in the form of a time average of a double-folding integral whose integrand involves functions that all have finite space extension, as the densities $\rho_k(\boldsymbol{r}_k)$ represent finite regions of space, and the radial range of the function $U(t, \boldsymbol{r})$ is given by $r = |\boldsymbol{r}| = t$ as it contains the delta function $\delta(t - r)$ and its derivatives. As such, the BR geometric factors are particularly well suited to evaluation by the means of Fourier–Bessel expansions [10].

To this end, the multipoles $U_{lm}(t, r)$ of the nonspherical function $U(t, \boldsymbol{r})$, defined by a multipole expansion

$$U(t, \boldsymbol{r}) = \sum_{lm} U_{lm}(t, r) \mathrm{i}^l Y_{lm}(\hat{\boldsymbol{r}}) \tag{13}$$

are expanded as Fourier–Bessel series in spherical Bessel functions $j_l(q_n^{(l)} r)$ in a range $0 \leqslant r < r_{\mathrm{ex}}$:

$$U_{lm}(t, r) = \sum_{n=1}^{\infty} c_{U\,n}^{(lm)}(t) j_l(q_n^{(l)} r) \tag{14}$$

where $q_n^{(l)} r_{\mathrm{ex}}$ are the positive roots of $j_l(x)$. The coefficients $c_{U\,n}^{(lm)}(t)$ in (14) are given in terms of the multipoles $U_{lm}(t, r)$ as

$$c_{U\,n}^{(lm)}(t) = \frac{1}{w_n^{(l)}} \int_0^{r_{\mathrm{ex}}} U_{lm}(t, r) j_l(q_n^{(l)} r) r^2 \,\mathrm{d}r \tag{15}$$

where

$$w_n^{(l)} = \frac{r_{\mathrm{ex}}}{2} [r_{\mathrm{ex}} j_l'(q_n^{(l)} r_{\mathrm{ex}})]^2. \tag{16}$$

The multipole expansion (13) can thus be written for $|\boldsymbol{r}| < r_{\mathrm{ex}}$ as

$$U(t, \boldsymbol{r}) = \frac{1}{4\pi} \sum_{lm} \sum_{n=1}^{\infty} c_{U\,n}^{(lm)}(t) \int \exp(\mathrm{i}\boldsymbol{q}_n^{(l)} \cdot \boldsymbol{r}) Y_{lm}(\hat{\boldsymbol{q}}_n^{(l)}) \,\mathrm{d}\hat{\boldsymbol{q}}_n^{(l)} \tag{17}$$

where $\boldsymbol{q}_n^{(l)}$ are vectors with polar angles $\hat{\boldsymbol{q}}_n^{(l)}$ and discrete moduli $|\boldsymbol{q}_n^{(l)}| = q_n^{(l)}$, and where the identity

$$j_l(qr) \mathrm{i}^l Y_{lm}(\hat{\boldsymbol{r}}) = \frac{1}{4\pi} \int \exp(\mathrm{i}\boldsymbol{q} \cdot \boldsymbol{r}) Y_{lm}(\hat{\boldsymbol{q}}) \,\mathrm{d}\hat{\boldsymbol{q}} \tag{18}$$

is employed.

In the double-folding integral of equation (9), the value of the function $U(t, \boldsymbol{r})$ is required only when

$$|\boldsymbol{r}| = |\boldsymbol{R} + \boldsymbol{r}_2 - \boldsymbol{r}_1| \leqslant r_{\max} = R + R_1 + R_2 \tag{19}$$

where R is the separation of the centres of the two densities, and R_1 and R_2 are the radii beyond which the uniform densities $\rho_1(\boldsymbol{r}_1)$ and $\rho_2(\boldsymbol{r}_2)$, respectively, vanish—this is simply because the product $\rho_1(\boldsymbol{r}_1)\rho_2(\boldsymbol{r}_2)$ in the integrand of (9) is bound to be zero when $|\boldsymbol{r}| > r_{\max}$. Thus, the validity of the Fourier–Bessel expansion (17) in the double-folding integral of equation (9) will be guaranteed when the expansion radius $r_{\mathrm{ex}} \geqslant r_{\max}$. Substituting then the expansion (17) with $\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{r}_2 - \boldsymbol{r}_1$ and an expansion radius $r_{\mathrm{ex}} \geqslant r_{\max}$ in equation (9), one obtains a Fourier–Bessel expansion for the BR geometric factor $\bar{C}_U^{(\mathrm{I,II})}$:

$$\bar{C}_U^{(\mathrm{I,II})} = \frac{1}{4\pi} \sum_{lm} \sum_{n=1}^{\infty} \bar{c}_{U\,n}^{(lm)} \int \tilde{\rho}_1(-\boldsymbol{q}_n^{(l)}) \tilde{\rho}_2(\boldsymbol{q}_n^{(l)}) \exp(\mathrm{i}\boldsymbol{q}_n^{(l)} \cdot \boldsymbol{R}) Y_{lm}(\hat{\boldsymbol{q}}_n^{(l)}) \,\mathrm{d}\hat{\boldsymbol{q}}_n^{(l)}. \tag{20}$$

Here,

$$\bar{c}_{U\,n}^{(lm)} = \frac{1}{\Delta t_1 \Delta t_2} \int_0^{\Delta t_1} \mathrm{d}t_1 \int_T^{T+\Delta t_2} \mathrm{d}t_2\, c_{U\,n}^{(lm)}(t) \tag{21}$$

are the time averages of the coefficients (15), and $\tilde{\rho}_1(-\boldsymbol{q}_n^{(l)})$ and $\tilde{\rho}_2(\boldsymbol{q}_n^{(l)})$ are the Fourier transforms of the densities $\rho_k(\boldsymbol{r}_k)$,

$$\tilde{\rho}_k(\boldsymbol{q}) = \int \rho_k(\boldsymbol{r}_k) \exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k)\, \mathrm{d}\boldsymbol{r}_k \tag{22}$$

evaluated at the points $\boldsymbol{q} = -\boldsymbol{q}_n^{(l)}$ and $\boldsymbol{q}_n^{(l)}$, respectively.

The Fourier transforms $\tilde{\rho}_k(\boldsymbol{q})$ can be expanded in multipoles also,

$$\tilde{\rho}_k(\boldsymbol{q}) = \sum_{l_k m_k} \tilde{\rho}_{l_k m_k}^{(k)}(q)\mathrm{i}^{l_k} Y_{l_k m_k}(\hat{\boldsymbol{q}}) \tag{23}$$

which are given in terms of the similarly defined multipoles $\rho_{l_k m_k}^{(k)}(r_k)$ of the densities $\rho_k(\boldsymbol{r}_k)$ themselves by

$$\tilde{\rho}_{l_k m_k}^{(k)}(q) = 4\pi \mathrm{i}^{l_k} \int_0^\infty \rho_{l_k m_k}^{(k)}(r_k)\, j_{l_k}(q r_k) r_k^2\, \mathrm{d}r_k. \tag{24}$$

When the multipole expansions (23) are substituted in (20), the products of spherical harmonics are expanded in terms of single spherical harmonics, and the identity (18) is used again, the Fourier–Bessel expansion of the BR geometric factor $\bar{C}_U^{(\mathrm{I,II})}$ can be written finally as†

$$\begin{aligned} \bar{C}_U^{(\mathrm{I,II})} = \frac{1}{4\pi} \sum_{lm} \sum_{\substack{l_1 m_1 \\ l_2 m_2}} \sum_{n=1}^{\infty} & \bar{c}_{U\,n}^{(lm)} (-1)^m \hat{l}\hat{l}_1\hat{l}_2 \mathrm{i}^{-l_1} \tilde{\rho}_{l_1 m_1}^{(1)}(q_n^{(l)}) \mathrm{i}^{l_2} \tilde{\rho}_{l_2 m_2}^{(2)}(q_n^{(l)}) \\ & \times \sum_{\lambda'\lambda} \hat{\lambda}'^2 \hat{\lambda} \begin{pmatrix} l_1 & l_2 & \lambda' \\ m_1 & m_2 & -\mu' \end{pmatrix} \begin{pmatrix} l_1 & l_2 & \lambda' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda' & l & \lambda \\ \mu' & m & -\mu \end{pmatrix} \\ & \times \begin{pmatrix} \lambda' & l & \lambda \\ 0 & 0 & 0 \end{pmatrix} j_\lambda(q_n^{(l)} R) \mathrm{i}^\lambda Y_{\lambda\mu}(\hat{\boldsymbol{R}}) \end{aligned} \tag{25}$$

where $\hat{l} = (2l+1)^{1/2}$ etc, and the large parenthesis denote 3-j coefficients; $\mu' = m_1 + m_2$ and $\mu = m + m_1 + m_2$. In this way, when the multipole expansions of the function $U(t, \boldsymbol{r})$ and the densities $\rho_k(\boldsymbol{r}_k)$ are given, the evaluation of the eight-dimensional integral (9) is reduced to the evaluation of the one-dimensional integrals (15) for $c_{U\,n}^{(lm)}(t)$ and (24) for $\tilde{\rho}_{l_k m_k}^{(k)}(q_n^{(l)})$, of the two-dimensional integrals (21) for the time averages $\bar{c}_{U\,n}^{(lm)}$, and of the Fourier–Bessel expansion (25), where, in principle, only the number of terms in the expansion controls the degree of approximation to the exact value of the BR geometric factor $\bar{C}_U^{(\mathrm{I,II})}$.

We now turn to the calculation of the Fourier–Bessel coefficients (15) and their time averages (21). It turns out that with the functional forms (10)–(12) taken by $U(t, \boldsymbol{r})$, these quantities can be evaluated in terms of elementary functions. Let us first determine the multipole expansion of the function $U_{A_{xx}}(t, \boldsymbol{r})$, given by equation (10). First, we regularize the space derivative part of $U_{A_{xx}}(t, \boldsymbol{r})$ by

$$\frac{\partial^2}{\partial x_1 \partial x_2} \frac{\delta(t-r)}{r} = \lim_{\epsilon\to 0} \frac{\partial^2}{\partial x_1 \partial x_2} \frac{\delta(t-r)}{r+\epsilon} \tag{26}$$

† Cf equation (21) of [10], which is reconciled with our equation (25) on noting that our displacement $\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{r}_2 - \boldsymbol{r}_1$ is defined there as $\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{r}_1 - \boldsymbol{r}_2$, and that $[\tilde{\rho}_{lm}(q)]^* = (-1)^m \tilde{\rho}_{l-m}(q)$ for a real density $\rho(\boldsymbol{r})$.

where the limit $\epsilon \to 0$ is understood to be taken only after a double integration. This yields

$$\frac{\partial^2}{\partial x_1 \partial x_2}\frac{\delta(t-r)}{r} = -\lim_{\epsilon\to 0}\left\{\left[\frac{\delta''(t-r)}{r+\epsilon} + \frac{3r+\epsilon}{r(r+\epsilon)^2}\delta'(t-r) + \frac{3r+\epsilon}{r(r+\epsilon)^3}\delta(t-r)\right]\right.$$
$$\left.\times\frac{(x_2-x_1)^2}{r^2} - \frac{\delta'(t-r)}{r(r+\epsilon)} - \frac{\delta(t-r)}{r(r+\epsilon)^2}\right\} \qquad (27)$$

while the time derivatives give

$$\frac{\partial^2}{\partial t_1 \partial t_2}\frac{\delta(t-r)}{r} = -\frac{\delta''(t-r)}{r}. \qquad (28)$$

Now

$$\frac{1}{r^2}(x_2-x_1)^2 = \frac{2\pi}{3}[Y_{1-1}(\hat{\boldsymbol{r}}) - Y_{11}(\hat{\boldsymbol{r}})]^2 = \frac{1}{3} + \sqrt{\frac{2\pi}{15}}Y_{2-2}(\hat{\boldsymbol{r}}) - \frac{1}{3}\sqrt{\frac{4\pi}{5}}Y_{20}(\hat{\boldsymbol{r}})$$
$$+\sqrt{\frac{2\pi}{15}}Y_{22}(\hat{\boldsymbol{r}}). \qquad (29)$$

Using this in (27), then substituting (27) and (28) in (10) and taking the limit $\epsilon \to 0$ everywhere except in the $\delta(t-r)$ term of the monopole component, we obtain the function $U_{A_{xx}}(t,\boldsymbol{r})$ as a multipole sum

$$U_{A_{xx}}(t,\boldsymbol{r}) = -\frac{2}{3}\sqrt{4\pi}\left[\frac{\delta''(t-r)}{r} + \lim_{\epsilon\to 0}\frac{\epsilon}{(r+\epsilon)^3}\frac{\delta(t-r)}{r}\right]\mathrm{i}^0 Y_{00}(\hat{\boldsymbol{r}}) - \sqrt{\frac{2\pi}{15}}\left[\frac{\delta''(t-r)}{r}\right.$$
$$\left. + 3\frac{\delta'(t-r)}{r^2} + 3\frac{\delta(t-r)}{r^3}\right]\mathrm{i}^2\left[Y_{2-2}(\hat{\boldsymbol{r}}) - \sqrt{\tfrac{2}{3}}Y_{20}(\hat{\boldsymbol{r}}) + Y_{22}(\hat{\boldsymbol{r}})\right]. \qquad (30)$$

The limit $\epsilon \to 0$ was performed in the multipole expansion of $U_{A_{xx}}(t,\boldsymbol{r})$ everywhere except in the $\delta(t-r)$ term of the monopole component because it can be seen easily that it is only with this term that the regularization (26) can contribute to the geometric factor $A_{xx}^{(\mathrm{I,II})}$. No regularization is required in the functions (11) and (12), which turn out not to contain any monopole componets:

$$U_{A_{xy}}(t,\boldsymbol{r}) = -\mathrm{i}\sqrt{\frac{2\pi}{15}}\left[\frac{\delta''(t-r)}{r} + 3\frac{\delta'(t-r)}{r^2} + 3\frac{\delta(t-r)}{r^3}\right]\mathrm{i}^2[Y_{2-2}(\hat{\boldsymbol{r}}) - Y_{22}(\hat{\boldsymbol{r}})] \qquad (31)$$

$$U_{B_{xy}}(t,\boldsymbol{r}) = \mathrm{i}\sqrt{\frac{4\pi}{3}}\left[\frac{\delta''(t-r)}{r} + \frac{\delta'(t-r)}{r^2}\right]\mathrm{i}Y_{10}(\hat{\boldsymbol{r}}). \qquad (32)$$

The multipoles $U_{lm}(r,t)$ of the various forms of $U(t,\boldsymbol{r})$ are found easily from equations (30)–(32), and the integrals required for the monopole ($l=0$), dipole ($l=1$) and quadrupole ($l=2$) Fourier–Bessel coefficients (15) are evaluated as follows:

$$\int_0^{r_{\mathrm{ex}}}\left[\frac{\delta''(t-r)}{r} + \lim_{\epsilon\to 0}\frac{\epsilon}{(r+\epsilon)^3}\frac{\delta(t-r)}{r}\right]j_0(qr)r^2\mathrm{d}r = -q\sin(qt)\Theta(t)\Theta(r_{\mathrm{ex}}-t)$$
$$+\Delta^{(0)}(q,r_{\mathrm{ex}},t) + \frac{1}{q}\lim_{\epsilon\to 0}\frac{\epsilon\sin(qt)}{(t+\epsilon)^3}\Theta(t)\Theta(r_{\mathrm{ex}}-t) \qquad (33)$$

$$\int_0^{r_{\mathrm{ex}}}\left[\frac{\delta''(t-r)}{r} + \frac{\delta'(t-r)}{r^2}\right]j_1(qr)r^2\mathrm{d}r = q\cos(qt)\Theta(t)\Theta(r_{\mathrm{ex}}-t) + \Delta^{(1)}(q,r_{\mathrm{ex}},t) \qquad (34)$$

$$\int_0^{r_{\mathrm{ex}}}\left[\frac{\delta''(t-r)}{r} + 3\frac{\delta'(t-r)}{r^2} + 3\frac{\delta(t-r)}{r^3}\right]j_2(qr)r^2\mathrm{d}r = q\sin(qt)\Theta(t)\Theta(r_{\mathrm{ex}}-t)$$
$$+\Delta^{(2)}(q,r_{\mathrm{ex}},t). \qquad (35)$$

Here,

$$\Delta^{(0)}(q, r_{\rm ex}, t) = \delta(t) - \cos(qr_{\rm ex})\delta(t - r_{\rm ex}) - r_{\rm ex} j_0(qr_{\rm ex})\delta'(t - r_{\rm ex}) \tag{36}$$

$$\Delta^{(1)}(q, r_{\rm ex}, t) = -\sin(qr_{\rm ex})\delta(t - r_{\rm ex}) - r_{\rm ex} j_1(qr_{\rm ex})\delta'(t - r_{\rm ex}) \tag{37}$$

$$\Delta^{(2)}(q, r_{\rm ex}, t) = [\cos(qr_{\rm ex}) - j_0(qr_{\rm ex}) - j_2(qr_{\rm ex})]\delta(t - r_{\rm ex}) - r_{\rm ex} j_2(qr_{\rm ex})\delta'(t - r_{\rm ex}). \tag{38}$$

These results are obtained following the rules that govern the use of the delta function and its derivatives†:

$$\int_{x_1}^{x_2} f(x)\delta(x - x_0)\,{\rm d}x = f(x_0)\Theta(x_0 - x_1)\Theta(x_2 - x_0) \tag{39}$$

$$\begin{aligned}\int_{x_1}^{x_2} f(x)\delta'(x - x_0)\,{\rm d}x = {}& -f'(x_0)\Theta(x_0 - x_1)\Theta(x_2 - x_0) + f(x_2)\delta(x_2 - x_0)\\ & - f(x_1)\delta(x_1 - x_0)\end{aligned} \tag{40}$$

$$\begin{aligned}\int_{x_1}^{x_2} f(x)\delta''(x - x_0)\,{\rm d}x = {}& f''(x_0)\Theta(x_0 - x_1)\Theta(x_2 - x_0) - f'(x_2)\delta(x_2 - x_0)\\ & + f'(x_1)\delta(x_1 - x_0) + f(x_2)\delta'(x_2 - x_0) - f(x_1)\delta'(x_1 - x_0)\end{aligned} \tag{41}$$

where $\Theta(t)$ is the Heaviside step function: $\Theta(t) = 1$ for $t > 0$, and $\Theta(t) = 0$ for $t < 0$; and utilizing the fact that $\lim_{x\to 0} j_l(x) = \lim_{x\to 0} {\rm d}[x j_l(x)]/{\rm d}x = \delta_{0l}$.

Using equations (30)–(35), the time-averaged Fourier–Bessel coefficients (21) needed in (25) for the BR geometric factor $\bar{A}^{(\rm I,II)}_{xx}$ are thus given by

$$\begin{aligned}\bar{c}^{(00)}_{A_{xx}n} = {}& \frac{2}{3}\sqrt{4\pi}\,\frac{1}{w^{(0)}_n}\Bigg[q^{(0)}_n\langle\sin(q^{(0)}_n t)\Theta(t)\Theta(r_{\rm ex} - t)\rangle - \langle\Delta^{(0)}(q^{(0)}_n, r_{\rm ex}, t)\rangle\\ & - \frac{1}{q^{(0)}_n}\lim_{\epsilon\to 0}\langle\epsilon(t+\epsilon)^{-3}\sin(q^{(0)}_n t)\Theta(t)\Theta(r_{\rm ex} - t)\rangle\Bigg]\end{aligned} \tag{42}$$

$$\bar{c}^{(2\pm 2)}_{A_{xx}n} = -\sqrt{\frac{3}{2}}\bar{c}^{(20)}_{A_{xx}n} = -\sqrt{\frac{2\pi}{15}}\,\frac{1}{w^{(2)}_n}[q^{(2)}_n\langle\sin(q^{(2)}_n t)\Theta(t)\Theta(r_{\rm ex} - t)\rangle + \langle\Delta^{(2)}(q^{(2)}_n, r_{\rm ex}, t)\rangle] \tag{43}$$

for the BR geometric factor $\bar{A}^{(\rm I,II)}_{xy}$ by

$$\bar{c}^{(2\pm 2)}_{A_{xy}n} = \pm{\rm i}\sqrt{\frac{2\pi}{15}}\,\frac{1}{w^{(2)}_n}[q^{(2)}_n\langle\sin(q^{(2)}_n t)\Theta(t)\Theta(r_{\rm ex} - t)\rangle + \langle\Delta^{(2)}(q^{(2)}_n, r_{\rm ex}, t)\rangle] \tag{44}$$

and for the BR geometric factor $\bar{B}^{(\rm I,II)}_{xy}$ by

$$\bar{c}^{(10)}_{B_{xy}n} = {\rm i}\sqrt{\frac{4\pi}{3}}\,\frac{1}{w^{(1)}_n}[q^{(1)}_n\langle\cos(q^{(1)}_n t)\Theta(t)\Theta(r_{\rm ex} - t)\rangle + \langle\Delta^{(1)}(q^{(1)}_n, r_{\rm ex}, t)\rangle]. \tag{45}$$

Here, the quantities $w^{(l)}_n$ are given by equation (16), and the angular brackets denote the time averaging

$$\langle f(t)\rangle \equiv \frac{1}{\Delta t_1 \Delta t_2}\int_0^{\Delta t_1}{\rm d}t_1\int_T^{T+\Delta t_2}{\rm d}t_2\, f(t_2 - t_1). \tag{46}$$

Let us first evaluate the time averages $\langle\sin(qt)\Theta(t)\Theta(r_{\rm ex}-t)\rangle$ and $\langle\cos(qt)\Theta(t)\Theta(r_{\rm ex}-t)\rangle$. A straightforward way of doing so is to integrate by parts in both t_2 and t_1, and to utilize the

† As a multidimensional integration with finite integration limits is involved here, care has to be taken to use the derivatives of the delta function properly.

fact that $\mathrm{d}\Theta(x)/\mathrm{d}x = \delta(x)$:

$$\begin{aligned}
&\int_0^{\Delta t_1} \mathrm{d}t_1 \int_T^{T+\Delta t_2} \mathrm{d}t_2 q\ \sin[q(t_2-t_1)]\Theta(t_2-t_1)\Theta(r_{\mathrm{ex}}-t_2+t_1) \\
&\quad= \int_0^{\Delta t_1} \Big\{ -\cos[q(t_2-t_1)]\Theta(t_2-t_1)\Theta(r_{\mathrm{ex}}-t_2+t_1)\big|_{t_2=T}^{T+\Delta t_2} \\
&\qquad + \int_T^{T+\Delta t_2} \cos[q(t_2-t_1)][\delta(t_2-t_1)\Theta(r_{\mathrm{ex}}-t_2+t_1) \\
&\qquad -\Theta(t_2-t_1)\delta(r_{\mathrm{ex}}-t_2+t_1)]\,\mathrm{d}t_2 \Big\}\,\mathrm{d}t_1 \\
&\quad= \frac{1}{q}\sin[q(T+\Delta t_2-t_1)]\Theta(T+\Delta t_2-t_1)\Theta(r_{\mathrm{ex}}-T-\Delta t_2+t_1)\big|_{t_1=0}^{\Delta t_1} \\
&\qquad -\frac{1}{q}\int_0^{\Delta t_1} \sin[q(T+\Delta t_2-t_1)]\Theta(T+\Delta t_2-t_1)\delta(r_{\mathrm{ex}}-T-\Delta t_2+t_1)\,\mathrm{d}t_1 \\
&\qquad -\frac{1}{q}\sin[q(T-t_1)]\Theta(T-t_1)\Theta(r_{\mathrm{ex}}-T+t_1)\big|_{t_1=0}^{\Delta t_1} \\
&\qquad +\frac{1}{q}\int_0^{\Delta t_1} \sin[q(T-t_1)]\Theta(T-t_1)\delta(r_{\mathrm{ex}}-T+t_1)\,\mathrm{d}t_1 \\
&\qquad + \int_0^{\Delta t_1} \{\Theta(t_1-T)\Theta(T+\Delta t_2-t_1) \\
&\qquad -\cos(qr_{\mathrm{ex}})\Theta(r_{\mathrm{ex}}-T+t_1)\Theta(T+\Delta t_2-r_{\mathrm{ex}}-t_1)\}\mathrm{d}t_1.
\end{aligned} \tag{47}$$

Here, terms with $\sin x\,\delta(x)$ were dropped immediately. This gives for the time average $\langle\sin(qt)\Theta(t)\Theta(r_{\mathrm{ex}}-t)\rangle$:

$$\begin{aligned}
&\Delta t_1\Delta t_2\, q\,\langle\sin(qt)\Theta(t)\Theta(r_{\mathrm{ex}}-t)\rangle \\
&\quad= \frac{1}{q}\{\sin[q(T+\Delta t_2-\Delta t_1)]\Theta(T+\Delta t_2-\Delta t_1)\Theta(r_{\mathrm{ex}}-T-\Delta t_2+\Delta t_1) \\
&\qquad -\sin[q(T+\Delta t_2)]\Theta(T+\Delta t_2)\Theta(r_{\mathrm{ex}}-T-\Delta t_2) \\
&\qquad -\sin[q(T-\Delta t_1)]\Theta(T-\Delta t_1)\Theta(r_{\mathrm{ex}}-T+\Delta t_1) \\
&\qquad +\sin(qT)\Theta(T)\Theta(r_{\mathrm{ex}}-T) \\
&\qquad -\sin(qr_{\mathrm{ex}})[\Theta(T+\Delta t_2-r_{\mathrm{ex}})\Theta(\Delta t_1-T-\Delta t_2+r_{\mathrm{ex}}) \\
&\qquad -\Theta(T-r_{\mathrm{ex}})\Theta(\Delta t_1-T+r_{\mathrm{ex}})]\} - \cos(qr_{\mathrm{ex}})\Theta(\Delta t_1-T+r_{\mathrm{ex}}) \\
&\qquad \times\Theta(T+\Delta t_2-r_{\mathrm{ex}})[\min(\Delta t_1, T+\Delta t_2-r_{\mathrm{ex}})-\max(T-r_{\mathrm{ex}},0)] \\
&\qquad +\Theta(\Delta t_1-T)\Theta(T+\Delta t_2)[\min(\Delta t_1, T+\Delta t_2)-\max(T,0)].
\end{aligned} \tag{48}$$

The time average $\langle\cos(qt)\Theta(t)\Theta(r_{\mathrm{ex}}-t)\rangle$, calculated in a way similar to that for the time average (48), yields

$$\begin{aligned}
&\Delta t_1\Delta t_2\, q\,\langle\cos(qt)\Theta(t)\Theta(r_{\mathrm{ex}}-t)\rangle \\
&\quad= \frac{1}{q}\{\cos[q(T+\Delta t_2-\Delta t_1)]\Theta(T+\Delta t_2-\Delta t_1)\Theta(r_{\mathrm{ex}}-T-\Delta t_2+\Delta t_1) \\
&\qquad -\cos[q(T+\Delta t_2)]\Theta(T+\Delta t_2)\Theta(r_{\mathrm{ex}}-T-\Delta t_2) \\
&\qquad -\cos[q(T-\Delta t_1)]\Theta(T-\Delta t_1)\Theta(r_{\mathrm{ex}}-T+\Delta t_1) \\
&\qquad +\cos(qT)\Theta(T)\Theta(r_{\mathrm{ex}}-T)-\cos(qr_{\mathrm{ex}})[\Theta(T+\Delta t_2-r_{\mathrm{ex}})
\end{aligned}$$

$$
\begin{aligned}
&\times\Theta(\Delta t_1 - T - \Delta t_2 + r_{\rm ex}) - \Theta(T - r_{\rm ex})\Theta(\Delta t_1 - T + r_{\rm ex})] \\
&+\Theta(T + \Delta t_2)\Theta(\Delta t_1 - \Delta t_2 - T) - \Theta(T)\Theta(\Delta t_1 - T)\} + \sin(q r_{\rm ex}) \\
&\times\Theta(\Delta t_1 - T + r_{\rm ex})\Theta(T + \Delta t_2 - r_{\rm ex}) \\
&\times[\min(\Delta t_1, T + \Delta t_2 - r_{\rm ex}) - \max(T - r_{\rm ex}, 0)]. \qquad (49)
\end{aligned}
$$

The ambiguity that may arise in these expressions when the argument of the step function vanishes is removed correctly by the definition $\Theta(0) = \frac{1}{2}$. The correctness of the somewhat lengthy analytical expressions (48) and (49) was checked by performing the two-dimensional integration numerically for cases with the time intervals $(0, \Delta t_1)$ and $(T, T + \Delta t_2)$ in all the possible logical relations of one to another.

For the time averages $\langle \Delta^{(l)}(q, r_{\rm ex}, t)\rangle$, we need the time averages $\langle \delta(t - r_{\rm ex})\rangle$ and $\langle \delta'(t - r_{\rm ex})\rangle$, which are evaluated easily to give

$$
\begin{aligned}
\Delta t_1 \Delta t_2 \, \langle \delta(t - r_{\rm ex})\rangle &= \Theta(\Delta t_1 - T + r_{\rm ex})\Theta(T + \Delta t_2 - r_{\rm ex}) \\
&\quad\times[\min(\Delta t_1, T + \Delta t_2 - r_{\rm ex}) - \max(T - r_{\rm ex}, 0)] \qquad (50) \\
\Delta t_1 \Delta t_2 \, \langle \delta'(t - r_{\rm ex})\rangle &= \Theta(T + \Delta t_2 - r_{\rm ex})\Theta(\Delta t_1 - T - \Delta t_2 + r_{\rm ex}) \\
&\quad-\Theta(T - r_{\rm ex})\Theta(\Delta t_1 - T + r_{\rm ex}). \qquad (51)
\end{aligned}
$$

These time averages vanish in the limit $r_{\rm ex} \to \infty$. The $\epsilon \to 0$ term in equation (42) can be evaluated by taking the limit $\epsilon \to 0$ already after one time integration, and the result is

$$
\begin{aligned}
&\frac{1}{q} \lim_{\epsilon \to 0} \langle \epsilon (t + \epsilon)^{-3} \sin(qt)\Theta(t)\Theta(r_{\rm ex} - t)\rangle \\
&\qquad = \frac{\Theta(\Delta t_1 - T)\Theta(T + \Delta t_2)}{2\Delta t_1 \Delta t_2}[\min(\Delta t_1, T + \Delta t_2) - \max(T, 0)] \qquad (52)
\end{aligned}
$$

which equals one-half of the q-independent term in (48), which, in turn, equals the average $\langle \delta(t - r_{\rm ex})\rangle$ with $r_{\rm ex} = 0$ of equation (50).

Using equations (48)–(52) and utilizing the fact that the quantities $q_n^{(l)} r_{\rm ex}$ are the roots of the spherical Bessel functions $j_l(x)$ and thus $j_l(q_n^{(l)} r_{\rm ex}) = 0$, the time-averaged Fourier–Bessel coefficients (42)–(45) are obtained finally as follows:

$$
\begin{aligned}
\bar{c}_{A_{xx}n}^{(00)} = \frac{2}{3} \frac{\sqrt{4\pi}}{\Delta t_1 \Delta t_2 w_n^{(0)} q_n^{(0)}} &\{\sin[q_n^{(0)}(T + \Delta t_2 - \Delta t_1)]\Theta(T + \Delta t_2 - \Delta t_1) \\
&\times\Theta(r_{\rm ex} - T - \Delta t_2 + \Delta t_1) - \sin[q_n^{(0)}(T + \Delta t_2)]\Theta(T + \Delta t_2) \\
&\times\Theta(r_{\rm ex} - T - \Delta t_2) - \sin[q_n^{(0)}(T - \Delta t_1)]\Theta(T - \Delta t_1)\Theta(r_{\rm ex} - T + \Delta t_1) \\
&+ \sin(q_n^{(0)} T)\Theta(T)\Theta(r_{\rm ex} - T) \\
&- \tfrac{1}{2} q_n^{(0)} \Theta(\Delta t_1 - T)\Theta(T + \Delta t_2)[\min(\Delta t_1, T + \Delta t_2) - \max(T, 0)]\} \qquad (53)
\end{aligned}
$$

$$
\begin{aligned}
\bar{c}_{A_{xx}n}^{(2\pm2)} = -\sqrt{\frac{3}{2}} \bar{c}_{A_{xx}n}^{(20)} = \pm \mathrm{i} \bar{c}_{A_{xy}n}^{(2\pm2)} = -\sqrt{\frac{2\pi}{15}} \frac{1}{\Delta t_1 \Delta t_2 w_n^{(2)} q_n^{(2)}}& \\
\times\{\sin[q_n^{(2)}(T + \Delta t_2 - \Delta t_1)]\Theta(T + \Delta t_2 - \Delta t_1)\Theta(r_{\rm ex} - T - \Delta t_2 + \Delta t_1)& \\
- \sin[q_n^{(2)}(T + \Delta t_2)]\Theta(T + \Delta t_2)\Theta(r_{\rm ex} - T - \Delta t_2)& \\
- \sin[q_n^{(2)}(T - \Delta t_1)]\Theta(T - \Delta t_1)\Theta(r_{\rm ex} - T + \Delta t_1) + \sin(q_n^{(2)} T)\Theta(T)& \\
\times\Theta(r_{\rm ex} - T) - \sin(q_n^{(2)} r_{\rm ex})\{\Theta(T + \Delta t_2 - r_{\rm ex})\Theta(\Delta t_1 - T - \Delta t_2 + r_{\rm ex})& \\
-\Theta(T - r_{\rm ex})\Theta(\Delta t_1 - T + r_{\rm ex}) + \Theta(\Delta t_1 - T + r_{\rm ex})\Theta(T + \Delta t_2 - r_{\rm ex})& \\
\times r_{\rm ex}^{-1}[\min(\Delta t_1, T + \Delta t_2 - r_{\rm ex}) - \max(T - r_{\rm ex}, 0)]\}& \\
+q_n^{(2)} \Theta(\Delta t_1 - T)\Theta(T + \Delta t_2)[\min(\Delta t_1, T + \Delta t_2) - \max(T, 0)]\}& \qquad (54)
\end{aligned}
$$

$$\bar{c}^{(10)}_{B_{xy}\,n} = \mathrm{i}\sqrt{\frac{4\pi}{3}}\frac{1}{\Delta t_1 \Delta t_2 w_n^{(1)} q_n^{(1)}}\{\cos[q_n^{(1)}(T+\Delta t_2-\Delta t_1)]$$
$$\times\Theta(T+\Delta t_2-\Delta t_1)\Theta(r_{\mathrm{ex}}-T-\Delta t_2+\Delta t_1)$$
$$-\cos[q_n^{(1)}(T+\Delta t_2)]\Theta(T+\Delta t_2)\Theta(r_{\mathrm{ex}}-T-\Delta t_2)$$
$$-\cos[q_n^{(1)}(T-\Delta t_1)]\Theta(T-\Delta t_1)\Theta(r_{\mathrm{ex}}-T+\Delta t_1)+\cos(q_n^{(1)}T)\Theta(T)$$
$$\times\Theta(r_{\mathrm{ex}}-T)-\cos(q_n^{(1)}r_{\mathrm{ex}})[\Theta(T+\Delta t_2-r_{\mathrm{ex}})\Theta(\Delta t_1-T-\Delta t_2+r_{\mathrm{ex}})$$
$$-\Theta(T-r_{\mathrm{ex}})\Theta(\Delta t_1-T+r_{\mathrm{ex}})]+\Theta(T+\Delta t_2)\Theta(\Delta t_1-\Delta t_2-T)$$
$$-\Theta(T)\Theta(\Delta t_1-T)\}. \tag{55}$$

Equations (25) and (53)–(55) furnish a general solution to the problem of finding Fourier–Bessel expansions of the representative BR geometric factors (6)–(8) with space regions specified by the multipoles (24) of their Fourier transforms. Obviously, the formalism developed can easily be used to give Fourier–Bessel expansions of all possible BR geometric factors, and not only the representatives (6)–(8), as long as their space regions have well-behaved multipole expansions.

3. BR geometric factors with spherical space regions

3.1. Fourier–Bessel expansions

Formula (25) gives a Fourier–Bessel expansion of the BR geometric factor $\bar{C}_U^{(\mathrm{I,II})}$ for the general case when the uniform densities $\rho_k(\boldsymbol{r}_k)$ as well as the function $U(t,\boldsymbol{r})$ are not spherically symmetric functions of space coordinates. Let us assume now that the densities are spherically symmetric with radii R_k, $\rho_k(\boldsymbol{r}_k) = \rho_k(r_k) = (3/4\pi R_k^3)\Theta(R_k - r_k)$. Such an assumption should not entail any serious loss in generality as space regions of practical relevance can be approximated by regions of spherical shape, but the main reason for considering spherical space regions is that it simplifies considerably the formulation of the problem and the actual calculations. With spherical space regions, only the $l_1 = l_2 = 0$ terms contribute in the general formula (25). Substituting further $\tilde{\rho}_{00}^{(k)}(q) = (4\pi)^{1/2}3j_1(qR_k)/qR_k$ for the monopole components of the Fourier transforms of the spherical uniform densities $\rho_k(r_k)$, equation (25) simplifies to

$$\bar{C}_U^{(\mathrm{I,II})} = \frac{9}{R_1R_2}\sum_{lm}\sum_{n=1}^{\infty}\frac{\bar{c}_{U\,n}^{(lm)}}{(q_n^{(l)})^2}j_1(q_n^{(l)}R_1)j_1(q_n^{(l)}R_2)j_l(q_n^{(l)}R)\mathrm{i}^lY_{lm}(\hat{\boldsymbol{R}}). \tag{56}$$

But more importantly, this assumption allows an alternative and simpler formulation based on a Fourier–Bessel expansion of one of the spherically symmetric densities, say $\rho_1(r_1)$, instead of the one based on the Fourier–Bessel expansion of the function $U(t,\boldsymbol{r})$.

Let us then expand the uniform density $\rho_1(r_1)$ as a Fourier–Bessel series in the spherical Bessel functions $j_0(q_nr_1)$ in a range $0 \leqslant r_1 < r_{\mathrm{ex}}$:

$$\rho_1(r_1) = \sum_{n=1}^{\infty}c_nj_0(q_nr_1). \tag{57}$$

Here, $q_n = n\pi/r_{\mathrm{ex}}$, and the coefficients c_n are given by

$$c_n = \frac{2}{r_{\mathrm{ex}}}\left(\frac{n\pi}{r_{\mathrm{ex}}}\right)^2\int_0^{r_{\mathrm{ex}}}\rho_1(r_1)j_0\left(\frac{n\pi}{r_{\mathrm{ex}}}r_1\right)r_1^2\,\mathrm{d}r_1 = \frac{3n}{2r_{\mathrm{ex}}^2R_1{}'}j_1\left(\frac{n\pi}{r_{\mathrm{ex}}}R_1{}'\right) \tag{58}$$

where $R_1{}' = \min(R_1, r_{\mathrm{ex}})$, with R_1 the density's radius. Since the value of the density $\rho_1(r_1)$ is needed in the multiple integral (9) that defines the BR geometric factor $\bar{C}_U^{(\mathrm{I,II})}$ only when

$$r_1 = |\boldsymbol{R}+\boldsymbol{r}_2-\boldsymbol{r}| \leqslant r_{1\,\max} = R+R_2+\max(T+\Delta t_2, 0) \tag{59}$$

where R is the separation of the centres of the two densities, and R_2 and $\max(T + \Delta t_2, 0)$ are respectively the radii beyond which the density $\rho_2(r_2)$ and the function $U(t, \boldsymbol{r})$ vanish, the expansion radius $r_{\rm ex}$ should satisfy the relation $r_{\rm ex} \geqslant r_{1\,\max}$. Substituting then the expansion (57) with an expansion radius $r_{\rm ex} \geqslant r_{1\,\max}$ in equation (9), and utilizing the identity (18) for $j_0(q_n r_1)$,

$$j_0(q_n r_1) = \frac{1}{4\pi}\int \exp(\mathrm{i}\boldsymbol{q}_n \cdot \boldsymbol{r}_1)\,\mathrm{d}\hat{\boldsymbol{q}}_n \tag{60}$$

with $\boldsymbol{r}_1 = \boldsymbol{R} + \boldsymbol{r}_2 - \boldsymbol{r}$, one obtains for the BR geometric factor $\bar{C}_U^{(\mathrm{I,II})}$ a Fourier–Bessel expansion:

$$\bar{C}_U^{(\mathrm{I,II})} = \frac{1}{4\pi}\sum_{n=1}^{\infty} c_n \tilde{\rho}_2(q_n)\int \bar{\tilde{U}}(-\boldsymbol{q}_n)\exp(\mathrm{i}\boldsymbol{q}_n \cdot \boldsymbol{R})\,\mathrm{d}\hat{\boldsymbol{q}}_n. \tag{61}$$

Here,

$$\tilde{\rho}_2(q_n) = 4\pi\int_0^{\infty}\rho_2(r_2)\,j_0(q_n r_2)r_2^2\mathrm{d}r_2 = 3\frac{j_1(q_n R_2)}{q_n R_2} \tag{62}$$

is the Fourier transform of the uniform density $\rho_2(r_2)$ with a radius R_2, and $\bar{\tilde{U}}(\boldsymbol{q})$ is the time-averaged Fourier transform of the function $U(t, \boldsymbol{r})$:

$$\bar{\tilde{U}}(\boldsymbol{q}) = \frac{1}{\Delta t_1 \Delta t_2}\int_0^{\Delta t_1}\mathrm{d}t_1\int_T^{T+\Delta t_2}\mathrm{d}t_2\int \mathrm{d}\boldsymbol{r}\,U(t, \boldsymbol{r})\exp(\mathrm{i}\boldsymbol{q} \cdot \boldsymbol{r}). \tag{63}$$

With the multipole expansion (13) of $U(t, \boldsymbol{r})$ and a further use of the identity (18), the time-averaged Fourier transform (63) can be written also as a multipole sum

$$\bar{\tilde{U}}(\boldsymbol{q}) = \sum_{lm}\bar{\tilde{U}}_{lm}(q)\,\mathrm{i}^l\,Y_{lm}(\hat{\boldsymbol{q}}) \tag{64}$$

where

$$\bar{\tilde{U}}_{lm}(q) = \frac{4\pi\mathrm{i}^l}{\Delta t_1 \Delta t_2}\int_0^{\Delta t_1}\mathrm{d}t_1\int_T^{T+\Delta t_2}\mathrm{d}t_2\int_0^{\infty} r^2\mathrm{d}r\,U_{lm}(t, r)\,j_l(qr). \tag{65}$$

Using the multipole expansion (64) and again the identity (18), the Fourier–Bessel expansion (61) of $\bar{C}_U^{(\mathrm{I,II})}$ now takes the form of a multipole sum:

$$\bar{C}_U^{(\mathrm{I,II})} = \sum_{lm}\sum_{n=1}^{\infty} c_n\tilde{\rho}_2(q_n)\bar{\tilde{U}}_{lm}(q_n)\,j_l(q_n R)Y_{lm}(\hat{\boldsymbol{R}}). \tag{66}$$

Substituting here for c_n and $\tilde{\rho}_2(q_n)$ from equations (58) and (62), respectively, and writing q_n explicitly as $n\pi/r_{\rm ex}$, one obtains finally

$$\bar{C}_U^{(\mathrm{I,II})} = \frac{9}{2\pi r_{\rm ex} R_1{}' R_2}\sum_{lm}\sum_{n=1}^{\infty} j_1\left(\frac{n\pi}{r_{\rm ex}}R_1{}'\right) j_1\left(\frac{n\pi}{r_{\rm ex}}R_2\right)\bar{\tilde{U}}_{lm}\left(\frac{n\pi}{r_{\rm ex}}\right) j_l\left(\frac{n\pi}{r_{\rm ex}}R\right) Y_{lm}(\hat{\boldsymbol{R}}) \tag{67a}$$

$$R_1{}' = \min(R_1, r_{\rm ex}) \qquad r_{\rm ex} \geqslant R + R_2 + \max(T + \Delta t_2, 0). \tag{67b}$$

Note that while different spherical Bessel function roots $q_n^{(l)}r_{\rm ex}$ and associated weights $w_n^{(l)}$ are required with the different multipolarities l of the function $U(t, \boldsymbol{r})$ in the formula (56) based on the Fourier–Bessel expansion of this function, here only the simple quantities $q_n^{(0)} = n\pi/r_{\rm ex}$ and $w_n^{(0)} = r_{\rm ex}^3/2(n\pi)^2$ are used for all these multipolarities. For this practical reason, when one of the densities is a spherically symmetric function, the formulation based on the Fourier–Bessel expansion (57) is preferable to an application of the general formula

(25) to such a case. Moreover, because the infinite-radius Fourier transform of the function $U(t,\boldsymbol{r})$ is now used, rather than the finite-radius transforms (33)–(35), the expressions for the time-averaged multipoles (65) of the Fourier transform will be simpler than those required in (42)–(45) for the coefficients $\bar{c}_{U\,n}^{(lm)}$ needed in the general formulation. It follows from equations (15), (21), (42)–(45), (50)–(52) and (65) that the quantities $\bar{\tilde{U}}_{lm}(n\pi/r_{\mathrm{ex}})$ needed in (67) for the BR geometric factor $\bar{A}_{xx}^{(\mathrm{I,II})}$ are given by

$$\bar{\tilde{U}}_{00}^{(A_{xx})}(q) = (4\pi)^{3/2}\tfrac{2}{3}[q\,\langle\sin(qt)\Theta(t)\rangle - \tfrac{3}{2}\langle\delta(t)\rangle] \tag{68}$$

$$\bar{\tilde{U}}_{2\pm2}^{(A_{xx})}(q) = -\sqrt{\frac{3}{2}}\bar{\tilde{U}}_{20}^{(A_{xx})}(q) = 4\pi\sqrt{\frac{2\pi}{15}}q\,\langle\sin(qt)\Theta(t)\rangle \tag{69}$$

for the BR geometric factor $\bar{A}_{xy}^{(\mathrm{I,II})}$ by

$$\bar{\tilde{U}}_{2\pm2}^{(A_{xy})}(q) = \mp\mathrm{i}4\pi\sqrt{\frac{2\pi}{15}}q\,\langle\sin(qt)\Theta(t)\rangle \tag{70}$$

and for the BR geometric factor $\bar{B}_{xy}^{(\mathrm{I,II})}$ by

$$\bar{\tilde{U}}_{10}^{(B_{xy})}(q) = -4\pi\sqrt{\frac{4\pi}{3}}q\,\langle\cos(qt)\Theta(t)\rangle. \tag{71}$$

Here, the time averages $\langle\sin(qt)\Theta(t)\rangle$ and $\langle\cos(qt)\Theta(t)\rangle$ are the limits $r_{\mathrm{ex}}\to\infty$ of the finite-radius time averages (48) and (49):

$$\begin{aligned}
&\Delta t_1\Delta t_2 q\langle\sin(qt)\Theta(t)\rangle\\
&\quad= \frac{1}{q}\{\sin[q(T+\Delta t_2-\Delta t_1)]\Theta(T+\Delta t_2-\Delta t_1) - \sin[q(T+\Delta t_2)]\\
&\qquad\times\Theta(T+\Delta t_2) - \sin[q(T-\Delta t_1)]\Theta(T-\Delta t_1) + \sin(qT)\Theta(T)\}\\
&\qquad+\Theta(\Delta t_1-T)\Theta(T+\Delta t_2)[\min(\Delta t_1, T+\Delta t_2) - \max(T,0)]
\end{aligned} \tag{72}$$

$$\begin{aligned}
&\Delta t_1\Delta t_2\, q\,\langle\cos(qt)\Theta(t)\rangle\\
&\quad= \frac{1}{q}\{\cos[q(T+\Delta t_2-\Delta t_1)]\Theta(T+\Delta t_2-\Delta t_1) - \cos[q(T+\Delta t_2)]\\
&\qquad\times\Theta(T+\Delta t_2) - \cos[q(T-\Delta t_1)]\Theta(T-\Delta t_1) + \cos(qT)\Theta(T)\\
&\qquad+\Theta(T+\Delta t_2)\Theta(\Delta t_1-\Delta t_2-T) - \Theta(T)\Theta(\Delta t_1-T)\}
\end{aligned} \tag{73}$$

and the quantity $\langle\delta(t)\rangle$ in (68) is the time average (50) with $r_{\mathrm{ex}}=0$, which also happens to equal the q-independent term in (72) and twice the $\epsilon\to 0$ term of equation (52).

The greatly simplified (when compared with those of the general formulation) equations (67)–(73) give the Fourier–Bessel expansions of the representative BR geometric factors (6)–(8) with spherical space regions; it is gratifying to note that it was possible to express the terms of these expansions in terms of elementary functions. In fact, it turns out that the BR geometric factors with spherical space regions can be evaluated in closed form, which will provide a means of testing the accuracy of the Fourier–Bessel results, but this is most easily accomplished starting with the standard Fourier-integral treatment of the multiple folding integrals involved.

3.2. *Closed-form evaluation*

Using the standard Fourier-integral method of calculating multiple folding integrals, a BR geometric factor $\bar{C}_U^{(\mathrm{I,II})}$ with spherical space regions $\rho_k(r_k)$,

$$\bar{C}_U^{(\mathrm{I,II})} = \frac{1}{\Delta t_1\Delta t_2}\int_0^{\Delta t_1}\mathrm{d}t_1\int_T^{T+\Delta t_2}\mathrm{d}t_2\int\rho_1(r_1)\mathrm{d}\boldsymbol{r}_1\int\rho_2(r_2)\,\mathrm{d}\boldsymbol{r}_2\,U(t,\boldsymbol{r}) \tag{74}$$

can be written as a multipole expansion, the multipoles of which are Fourier integrals:

$$\bar{C}_U^{(\mathrm{I,II})} = \frac{4\pi}{(2\pi)^3} \sum_{lm} \int_0^\infty \tilde{\rho}_1(q)\tilde{\rho}_2(q)\bar{\tilde{U}}_{lm}(q)\, j_l(qR)q^2\mathrm{d}q\, Y_{lm}(\hat{\boldsymbol{R}}). \tag{75}$$

Here, $\bar{\tilde{U}}_{lm}(q)$ are the multipoles (65) of the time-averaged Fourier transform of the function $U(t, \boldsymbol{r})$, given by equations (68)–(73), and

$$\tilde{\rho}_k(q) = 4\pi \int_0^\infty \rho_k(r_k)\, j_0(qr_k)r_k^2\mathrm{d}r_k = 3\frac{j_1(qR_k)}{qR_k} \tag{76}$$

are the Fourier transforms of the spherically symmetric uniform densities $\rho_k(r_k)$, which have radii R_k and are normalized to unit volume. This result follows from the convolution theorem (see, for example, [12]) according to which the Fourier transform of a folding integral, like the one in equation (74), is the product of the Fourier transforms of the functions appearing in the folding integral; equation (75) is simply the time average of the Fourier transformation of the folding integral from the momentum space back to the configuration space. In general, Fourier integrals of the type of those in equation (75) have to be evaluated numerically, and the great advantage of the Fourier–Bessel formulation is that it replaces such numerical integration by the analytical method of a series expansion. However, in the case of spherical space regions with the Fourier transforms (76), and with the time-averaged Fourier transforms (68)–(73) of the functions $U(t, \boldsymbol{r})$, the Fourier integrals in (75) can in fact be evaluated in closed form, and we present such evaluation in this section.

The quantities $\bar{\tilde{U}}_{lm}(q)$ in (75), as given by equations (68)–(73), are linear combinations of terms of the form $\tau j_0(\tau q)$ and a q-independent term for the multipolarities $l = 0, 2$, and of terms of the form $\tau j_{-1}(\tau q)$ and a $1/q$ term for the multipolarity $l = 1$, while the Fourier transforms (76) of the densities are of the form $j_1(aq)/aq$. We define therefore the integrals

$$\mathrm{ji}_4(n; l_1, l_2, l_3, l_4; \alpha, \beta, \gamma, \delta) = \int_0^\infty j_{l_1}(\alpha x)\, j_{l_2}(\beta x)\, j_{l_3}(\gamma x)\, j_{l_4}(\delta x)\frac{\mathrm{d}x}{x^n} \tag{77}$$

the evaluation of which is required in (75) for the parameter values (i) $n = 0$, $l_1 = l_2 = 1$, $l_3 = 0$, $l_4 = 0, 2$; (ii) $n = 0$, $l_1 = l_2 = 1$, $l_3 = -1$, $l_4 = 1$; and, on account of the $1/q$ term in (73), (iii) $n = 1$, $l_1 = l_2 = 1$, $l_3 = 0$ (with $\gamma = 0$), $l_4 = 1$.

In principle, it should be possible to evaluate the integral $\mathrm{ji}_4(n; l_1, l_2, l_3, l_4; \alpha, \beta, \gamma, \delta)$ in closed form for all the integer values n and l_1, l_2, l_3, l_4 for which the integral exists, as the integrand can be written as a sum of a finite number of terms, with each term having the form $a\sin(bx)/x^k$ or $a\cos(bx)/x^k$, where k is an integer, and the indefinite integrals of such terms can be done in terms of the sine or cosine integrals. In practice, however, such a procedure is prohibitively lengthy for all but very small values of the parameters n, l_1, l_2, l_3, l_4: for example, the integrand of the integral $\mathrm{ji}_4(0; 1, 1, 0, 2; \alpha, \beta, \gamma, \delta)$ has already 96 terms of the above-mentioned form with k ranging from four to eight, and the indefinite integral of each of these terms generates in turn k new terms, giving in total 572 terms. Furthermore, complications arise when the limit $x \to 0$ (or $x \to \infty$, depending on the definition of the sine integral) is taken in the sine integrals, which have arguments of the form ax, as the signs of the parameters a, which are linear combinations of the parameters α, β, γ and δ, must be determined suitably. Remarkably, the computing system *Mathematica* [13] is able to perform the definite integrations in the integrals $\mathrm{ji}_4(n; l_1, l_2, l_3, l_4; \alpha, \beta, \gamma, \delta)$ required directly. After considerable simplifications, and using an economic way of writing linear combinations that contain terms with permuting signs by means of the definitions

$$\alpha_n = \alpha \qquad \beta_n = (-1)^n\beta \qquad \gamma_n = (-1)^{[n/2]}\gamma \qquad \delta_n = (-1)^{[n/4]}\delta$$
$$\delta'_n = (-1)^{[n/2]}\delta \tag{78}$$

where $[x]$ is the integer part of x, the results are as follows:

$$\mathrm{ji}_4(0;1,1,0,0;\alpha,\beta,\gamma,\delta)=\frac{\pi}{1920\alpha\beta}\sum_{n=0}^{7}\frac{|\alpha_n+\beta_n+\gamma_n+\delta_n|^3}{\alpha_n\beta_n\gamma_n\delta_n}\times[4\alpha_n^2+(4\beta_n-\gamma_n-\delta_n)(-3\alpha_n+\beta_n+\gamma_n+\delta_n)] \tag{79}$$

$$\mathrm{ji}_4(0;1,1,0,2;\alpha,\beta,\gamma,\delta)=-\frac{\pi}{26\,880\alpha\beta\delta^2}\sum_{n=0}^{7}\frac{|\alpha_n+\beta_n+\gamma_n+\delta_n|^3}{\alpha_n\beta_n\gamma_n\delta_n}\times\{(\alpha_n+\beta_n+\gamma_n)(\alpha_n+\beta_n+\gamma_n-3\delta_n)[6\alpha_n^2+(6\beta_n-\gamma_n)\times(-5\alpha_n+\beta_n+\gamma_n)]+[8(\alpha_n^2-12\alpha_n\beta_n+\beta_n^2)+9(\alpha_n+\beta_n)\gamma_n+\gamma_n^2]\delta_n^2+[24(\alpha_n+\beta_n)-11\gamma_n]\delta_n^3-8\delta_n^4\} \tag{80}$$

$$\mathrm{ji}_4(0;1,1,-1,1;\alpha,\beta,\gamma,\delta)=-\frac{\pi}{11\,520\alpha\beta\gamma\delta}\sum_{n=0}^{7}\frac{|\alpha_n+\beta_n+\gamma_n+\delta_n|^3}{\alpha_n\beta_n\delta_n}\times\{5\alpha_n^3-3\alpha_n^2(5\beta_n-3\gamma_n+5\delta_n)+(-3\alpha_n+\beta_n+\gamma_n+\delta_n)\times[5\beta_n^2+(\gamma_n-5\delta_n)(4\beta_n-\gamma_n-\delta_n)]\}. \tag{81}$$

Here, it is assumed that the parameters α, β, γ and δ are all nonzero; the required integrals that have some of these parameters equal to zero were evaluated separately:

$$\mathrm{ji}_4(0;1,1,0,0;\alpha,\beta,0,0)=\frac{\pi}{12\alpha\beta}\sum_{n=0}^{1}\frac{|\alpha_n+\beta_n|}{\alpha_n\beta_n}(\alpha_n^2-\alpha_n\beta_n+\beta_n^2) \tag{82}$$

$$\mathrm{ji}_4(0;1,1,0,0;\alpha,\beta,\gamma,0)=\frac{\pi}{192\alpha\beta}\sum_{n=0}^{3}\frac{(\alpha_n+\beta_n+\gamma_n)|\alpha_n+\beta_n+\gamma_n|}{\alpha_n\beta_n\gamma_n}\times[3(\alpha_n-\beta_n)^2+2(\alpha_n+\beta_n)\gamma_n-\gamma_n^2] \tag{83}$$

$$\mathrm{ji}_4(0;1,1,0,2;\alpha,\beta,0,\delta)=-\frac{\pi}{384\alpha\beta\delta^2}\sum_{n=0}^{3}\frac{(\alpha_n+\beta_n+\delta'_n)|\alpha_n+\beta_n+\delta'_n|}{\alpha_n\beta_n\delta'_n}\times(\alpha_n+\beta_n-\delta'_n)^2(\alpha_n^2-4\alpha_n\beta_n+\beta_n^2-\delta'^2_n) \tag{84}$$

$$\mathrm{ji}_4(1;1,1,0,1;\alpha,\beta,0,\delta)=-\frac{\pi}{1152\alpha\beta\delta}\sum_{n=0}^{3}\frac{|\alpha_n+\beta_n+\delta'_n|^3}{\alpha_n\beta_n\delta'_n}\times[\alpha_n^3-3\alpha_n(\beta_n^2-4\beta_n\delta'_n+\delta'^2_n)+(\beta_n+\delta'_n)(-3\alpha_n^2+\beta_n^2-4\beta_n\delta'_n+\delta'^2_n)]. \tag{85}$$

Here, the definition $j_0(0)=1$ is assumed in the integrals (77); as $\lim_{x\to0} j_l(x)=0$ when $l>0$, the integrals $\mathrm{ji}_4(n;1,1,l_3,l_4;\alpha,\beta,\gamma,0)$ with $l_4>0$ vanish. The integral $\mathrm{ji}_4(0;1,1,0,0;\alpha,\beta,0,\delta)$ that is required also is given already by the integral (83):

$$\mathrm{ji}_4(0;1,1,0,0;\alpha,\beta,0,\delta)=\mathrm{ji}_4(0;1,1,0,0;\alpha,\beta,\delta,0). \tag{86}$$

Using equations (68)–(73), (75) and (76), the integrals (79)–(86), and the definitions

$$a_{00}^{(A_{xx})}=(4\pi)^{3/2}\frac{2}{3}\qquad a_{2\pm1}^{(A_{xx})}=0\qquad a_{2\pm2}^{(A_{xx})}=-\sqrt{\frac{3}{2}}\,a_{20}^{(A_{xx})}=4\pi\sqrt{\frac{2\pi}{15}} \tag{87}$$

$$a_{20}^{(A_{xy})}=a_{2\pm1}^{(A_{xy})}=0\qquad a_{2\pm2}^{(A_{xy})}=\mp\mathrm{i}4\pi\sqrt{\frac{2\pi}{15}}\qquad b_{10}^{(B_{xy})}=-4\pi\sqrt{\frac{4\pi}{3}} \tag{88}$$

$$\tau_0=0\qquad \tau_1=T+\Delta t_2-\Delta t_1\qquad \tau_2=T+\Delta t_2\qquad \tau_3=T\qquad \tau_4=T-\Delta t_1 \tag{89}$$

$$g_0^{(0)}=-\frac{\Theta(-\tau_4)\Theta(\tau_2)}{2\Delta t_1\Delta t_2}[\min(\Delta t_1,\tau_2)-\max(\tau_3,0)] \tag{90}$$

$$g_0^{(1)} = \frac{1}{\Delta t_1 \Delta t_2}[\Theta(\tau_2)\Theta(-\tau_1) - \Theta(\tau_3)\Theta(-\tau_4)] \tag{91}$$

$$g_0^{(2)} = \frac{\Theta(-\tau_4)\Theta(\tau_2)}{\Delta t_1 \Delta t_2}[\min(\Delta t_1, \tau_2) - \max(\tau_3, 0)] \tag{92}$$

$$g_i^{(l)} = (-1)^{i+1}\frac{\Theta(\tau_i)}{\Delta t_1 \Delta t_2}[\tau_i + \mathrm{zr}(\tau_i)\delta_{1l}] \qquad l = 0, 1, 2, \quad i = 1, 2, 3, 4 \tag{93}$$

where

$$\mathrm{zr}(x) = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } x \neq 0 \end{cases} \tag{94}$$

and the definition $\Theta(0) = \frac{1}{2}$ is used again, we obtain finally the following closed-form expressions for the representative BR geometric factors (6)–(8) with spherical space regions:

$$\bar{A}_{xx}^{(\mathrm{I,II})} = \frac{4\pi}{(2\pi)^3}\frac{9}{R_1 R_2}\sum_{\substack{l=0,2\\ m}} a_{lm}^{(A_{xx})}\sum_{i=0}^{4} g_i^{(l)}\mathrm{ji}_4(0; 1, 1, 0, l; R_1, R_2, \tau_i, R)Y_{lm}(\hat{\boldsymbol{R}}) \tag{95}$$

$$\bar{A}_{xy}^{(\mathrm{I,II})} = \frac{4\pi}{(2\pi)^3}\frac{9}{R_1 R_2}\sum_{m} a_{2m}^{(A_{xy})}\sum_{i=0}^{4} g_i^{(2)}\,\mathrm{ji}_4(0; 1, 1, 0, 2; R_1, R_2, \tau_i, R)Y_{2m}(\hat{\boldsymbol{R}}) \tag{96}$$

$$\bar{B}_{xy}^{(\mathrm{I,II})} = \frac{4\pi}{(2\pi)^3}\frac{9}{R_1 R_2} b_{10}^{(B_{xy})}\sum_{i=0}^{4} g_i^{(1)}\mathrm{ji}_4[\mathrm{zr}(\tau_i); 1, 1, \mathrm{zr}(\tau_i)-1, 1; R_1, R_2, \tau_i, R]Y_{10}(\hat{\boldsymbol{R}}). \tag{97}$$

4. Numerical results and concluding remarks

The convergence properties of the Fourier–Bessel expansions were examined in numerical calculations of some representative examples of BR geometric coefficients with spherical space regions, using the formula (67) with the Fourier–Bessel coefficients given by equations (68)–(73) and an expansion radius $r_{\mathrm{ex}} = R + R_2 + \max(T + \Delta t_2, 0)$.

For the BR geometric factors to have an appreciable magnitude, it is obvious that, in a system of units where the speed of light $c = 1$, the separation in space and time of the space-time regions must not be much greater than the dimensions of the space-time regions themselves. The calculations were done using an unspecified unit of length; choosing the millimetre as the unit of length, for example, the unit of time equals approximately 3.33×10^{-12} s in a system of units with $c = 1$. This illustrates the fact that, on a realistic laboratory scale, the time intervals corresponding to even relatively large distances are very short, but we leave aside the question of how field measurements occupying and/or separated by such short time intervals can be realized.

The numerical values of the representative BR geometric factors (6)–(8) for various spherical space-time regions with dimensions of the order of unity and similar or smaller space-time separations, in a $c = 1$ system of units, are collected in table 1. As the quantities that are required in the field commutation relations are the absolute values of the differences $\bar{C}_U^{(\mathrm{I,II})} - \bar{C}_U^{(\mathrm{II,I})}$, while some BR spring constants require the value of the sum $\bar{C}_U^{(\mathrm{I,II})} + \bar{C}_U^{(\mathrm{II,I})}$, the 'reverse' geometric factor $\bar{C}_U^{(\mathrm{II,I})}$ was calculated together with a given geometric factor $\bar{C}_U^{(\mathrm{I,II})}$. This was done by interchanging the radii R_1 and R_2 of the spherical space regions and changing their relative displacement $\boldsymbol{R}$ to $-\boldsymbol{R}$ (i.e., changing the polar angles θ and ϕ of $\boldsymbol{R}$ to $\pi - \theta$ and $\phi + \pi$), together with interchanging the time intervals Δt_1 and Δt_2 and changing their separation T to $-T$. BR geometric factors $\bar{A}_{xx}^{(\mathrm{I,I})}$ with fully coinciding space-time regions were calculated also as they are required for some of the BR spring constants. When the sphere separation $R = 0$, only the monopole ($l = 0$) term contributes in the expansion (67) because $j_l(q_n R) \to 0$ for $l > 0$ as $R \to 0$, and thus the BR factors $\bar{A}_{xy}^{(\mathrm{I,II})}$ and $\bar{B}_{xy}^{(\mathrm{I,II})}$ with spherical space regions vanish when $R = 0$. The Fourier–Bessel expansions of the BR geometric factors, except those of the geometric factors $A_{xx}^{(\mathrm{I,II})}$ for spherical space regions whose centres coincide, converge rapidly as less than a hundred terms were required for the four-digit accuracy with which the geometric factors are printed in table 1.

The slow convergence of expansion (67) in the case of spherical space regions with coinciding centres is due to the presence of a q-independent term in the quantity $\bar{\bar{U}}_{00}^{(A_{xx})}(q)$ of equation (68). However, the contribution of this term to the BR geometric factor $\bar{A}_{xx}^{(\mathrm{I,II})}$ has then a simple form and can be summed easily. With sphere separation $R = 0$, it follows from equations (67) and (68) that the q-independent term contributes to the BR factor $\bar{A}_{xx}^{(\mathrm{I,II})}$ the quantity

$$\bar{A}_{xx}^{(\mathrm{I,II})}(g_0^{(0)}) = \frac{12 g_0^{(0)}}{\pi R_1{}' R_2} \frac{\pi}{r_{\mathrm{ex}}} \sum_{n=1}^{\infty} j_1\left(\frac{n\pi}{r_{\mathrm{ex}}} R_1{}'\right) j_1\left(\frac{n\pi}{r_{\mathrm{ex}}} R_2\right) \tag{98}$$

where $g_0^{(0)}$, defined by equation (90), is the q-independent term in question. The series in (98) is a Fourier–Bessel representation of the simplest of the integrals evaluated in section 3.2:

$$\begin{aligned} \frac{\pi}{r_{\mathrm{ex}}} \sum_{n=1}^{\infty} j_1\left(\frac{n\pi}{r_{\mathrm{ex}}} R_1{}'\right) j_1\left(\frac{n\pi}{r_{\mathrm{ex}}} R_2\right) &= \int_0^{\infty} j_1(R_1{}' q) j_1(R_2 q)\,\mathrm{d}q \\ &= \mathrm{ji}_4(0; 1, 1, 0, 0; R_1{}', R_2, 0, 0) = \frac{R_<}{R_>^2} \frac{\pi}{6}. \end{aligned} \tag{99}$$

Here, the result (82) is simplified using $R_<$ and $R_>$, which are respectively the lesser and the greater of the radii $R_1{}'$ and R_2, and the parameter r_{ex} should be such that $r_{\mathrm{ex}} \geqslant R_>$, which condition is guaranteed by that of equation (67*b*). When the result (99) is used in the expansion (67) in the cases of sphere separation $R = 0$, the rate of convergence improves dramatically and becomes similar to that of the geometric factors for space regions with noncoinciding centres.

The accuracy of the numerical results of table 1, obtained using the Fourier–Bessel expansions, was checked using the closed-form expressions (95)–(97). As no calculated values of the BR geometric factors could be found in the literature, this is the only check of the correctness and accuracy of our results. Admittedly, this check is not a fully independent one, as both the closed-form expressions and Fourier–Bessel expansions are obtained using Fourier-transform methods. However, the calculations reported here use only proven analytical methods, of which Fourier integrals and Fourier–Bessel expansions are a part, and thus our results would be invalidated only if the same algebraic errors were made in the derivation of the closed-form and Fourier–Bessel expressions. In this connection, we note that the analytical expressions (48) and (49) for the double time averages, used in both the Fourier-integral and Fourier–Bessel formulations, were checked by doing the two-dimensional integrations involved numerically.

A rather interesting result of the calculations is that the geometric factors $\bar{A}_{xx}^{(\mathrm{I,I})}$ with fully coinciding space-time regions turn out to have negative values. The closed-form expression (95) for the geometric factor $\bar{A}_{xx}^{(\mathrm{I,II})}$ with radii $R_1 = R_2 = R_0$, time intervals $\Delta t_1 = \Delta t_2 = \Delta t_0$ and separations $R = T = 0$ simplifies to†

$$\bar{A}_{xx}^{(\mathrm{I,I})} = -\frac{1}{8 R_0^4 \kappa}(4 + \kappa)(2 - \kappa)^2 \Theta(2 - \kappa) - \frac{1}{R_0^4 \kappa} \tag{100}$$

† Using this simple expression, it can be shown that, contrary to a conclusion of [8], the BR average 'self-force' on the field measurement's test body approximates correctly the average self-force that obtains when the duration of the momentum measurements on a test body of sufficiently great mass is sufficiently short [9].

Table 1. Representative BR geometric factors $\bar{C}_U^{(\mathrm{I,II})}$ for space-time regions I and II specified by space spheres of radii R_1 and R_2, and time intervals Δt_1 and Δt_2, respectively, with the centre of the second sphere displaced from that of the first one by a vector of spherical coordinates R, θ, ϕ, and the beginning of the second time interval separated from that of the first one by a time interval T; units such that the speed of light $c = 1$ are used.

	R_1	R_2	R	θ	ϕ	Δt_1	Δt_2	T	$\bar{C}_U^{(\mathrm{I,II})}$
$\bar{A}_{xx}^{(\mathrm{I,II})}$	1	1	0	–	–	1	1	0	$-1.625 \times 10^{+0}$
$\bar{A}_{xx}^{(\mathrm{I,II})}$	10	10	0	–	–	1	1	0	-2.850×10^{-3}
$\bar{A}_{xx}^{(\mathrm{I,II})}$	1	1	0	–	–	1	2	0.5	1.953×10^{-1}
$\bar{A}_{xx}^{(\mathrm{I,II})}$ [a]	1	1	0	–	–	2	1	−0.5	-5.664×10^{-1}
$\bar{A}_{xx}^{(\mathrm{I,II})}$	1	1	1	$\frac{1}{6}\pi$	$\frac{1}{3}\pi$	1	1	0.5	-6.407×10^{-2}
$\bar{A}_{xx}^{(\mathrm{I,II})}$ [a]	1	1	1	$\frac{5}{6}\pi$	$\frac{4}{3}\pi$	1	1	−0.5	-4.530×10^{-1}
$\bar{A}_{xy}^{(\mathrm{I,II})}$	1	1	1	$\frac{1}{6}\pi$	$\frac{1}{3}\pi$	1	1	0.5	6.636×10^{-2}
$\bar{A}_{xy}^{(\mathrm{I,II})}$ [a]	1	1	1	$\frac{5}{6}\pi$	$\frac{4}{3}\pi$	1	1	−0.5	5.901×10^{-3}
$\bar{B}_{xy}^{(\mathrm{I,II})}$	1	1	1	$\frac{1}{6}\pi$	$\frac{1}{3}\pi$	1	1	0.5	-2.730×10^{-1}
$\bar{B}_{xy}^{(\mathrm{I,II})}$ [a]	1	1	1	$\frac{5}{6}\pi$	$\frac{4}{3}\pi$	1	1	−0.5	1.675×10^{-1}
$\bar{A}_{xx}^{(\mathrm{I,II})}$	1	2	1	$\frac{1}{6}\pi$	$\frac{1}{3}\pi$	1	2	0.5	7.454×10^{-2}
$\bar{A}_{xx}^{(\mathrm{I,II})}$ [a]	2	1	1	$\frac{5}{6}\pi$	$\frac{4}{3}\pi$	2	1	−0.5	-8.914×10^{-2}
$\bar{A}_{xy}^{(\mathrm{I,II})}$	1	2	1	$\frac{1}{6}\pi$	$\frac{1}{3}\pi$	1	2	0.5	3.493×10^{-3}
$\bar{A}_{xy}^{(\mathrm{I,II})}$ [a]	2	1	1	$\frac{5}{6}\pi$	$\frac{4}{3}\pi$	2	1	−0.5	-3.884×10^{-4}
$\bar{B}_{xy}^{(\mathrm{I,II})}$	1	2	1	$\frac{1}{6}\pi$	$\frac{1}{3}\pi$	1	2	0.5	-2.560×10^{-2}
$\bar{B}_{xy}^{(\mathrm{I,II})}$ [a]	2	1	1	$\frac{5}{6}\pi$	$\frac{4}{3}\pi$	2	1	−0.5	4.126×10^{-3}

[a] The 'reverse' geometric factor $\bar{C}_U^{(\mathrm{II,I})}$ of the preceding entry.

where $\kappa = \Delta t_0/R_0$. For a fixed value of R_0, this function of the ratio κ increases monotonically from $-\infty$ when $\kappa \to 0$ to the value of zero for $\kappa \to \infty$. For $\kappa \geqslant 2$, the geometric factor $\bar{A}_{xx}^{(\mathrm{I,I})}$ reduces to the value $-1/R_0^4\kappa$, and so the BR average 'self-force' [1] on the field measurement's test body of charge density ρ_I is then $\rho_\mathrm{I}^2 V_\mathrm{I}^2 \Delta t_0 D_x^{(\mathrm{I})} \bar{A}_{xx}^{(\mathrm{I,I})} = -\rho_\mathrm{I}^2 V_\mathrm{I}^2 D_x^{(\mathrm{I})}/R_0^3$, which is simply the electrostatic force of attraction between the test and neutralizing bodies when their centres are displaced by a distance $|D_x^{(\mathrm{I})}| \ll R_0$. The negativity of the BR geometric factor $\bar{A}_{xx}^{(\mathrm{I,I})}$ means that the spring constant $k_\mathrm{I} = \rho_\mathrm{I}^2 V_\mathrm{I}^2 T_\mathrm{I} \bar{A}_{xx}^{(\mathrm{I,I})}$ of the spring that is used in a BR field measurement involving a space-time region $V_\mathrm{I}, T_\mathrm{I}$ to compensate the test body's average 'self-force' has to be negative. While it is certainly possible to envisage a spring mechanism that would provide a force proportional to and in the direction of a test body's displacement†, we note that Bohr and Rosenfeld did not consider it necessary to make a comment on this rather unusual specification that their measurement procedure would place on the spring mechanism—but one can now only speculate whether Bohr and Rosenfeld were in fact aware of this consequence of their analysis‡. In any case, despite its inherent instability, a spring mechanism with negative spring constant should present no difficulty of principle for a BR measurement procedure because a

† A spring system with a negative spring constant can be constructed as follows. Two elastically compressible rods, each of spring constant k and length $l + d$ when not stressed, are aligned 'head to tail' along a common axis and joined via a movable joint, while their outer ends are fastened using similar joints to rigid supports so that this spring system is compressed to a total length $2l$. It can be shown easily that a body attached to the middle joint and moved a small distance x, $x \ll l$, $x \ll d$, from the system's axis and in a direction perpendicular to it, will experience a force $F = (2kd/l)x = -\kappa x$, which is proportional to and acting in the direction of the displacement x, i.e., the spring constant $\kappa = -2kd/l$ of such a system is negative.

‡ A hint that Bohr and Rosenfeld were aware of the possibility of the geometric factor $\bar{A}_{xx}^{(\mathrm{I,I})}$ being negative is given by their careful writing of its square root as $|\bar{A}_{xx}^{(\mathrm{I,I})}|^{1/2}$.

BR spring, together with the test body to which it is attached, is supposed to be released only for the exact duration of the field measurement, and the spring force is designed so that its effect is compensated by the test body's 'self-force'.

We conclude that a well-controlled method for the computation of the BR geometric factors was developed using Fourier–Bessel expansions. The efficiency and accuracy of the method were tested numerically in the case of spherical space regions when a BR geometric factor can be represented by a Fourier–Bessel series with terms expressed entirely in terms of elementary functions, and it is possible to obtain the factor in terms of manageable closed-form expressions. The space-time-averaged electromagnetic-field commutators, as well as the formal expressions and '*gedanken'* experimental procedures of the famous BR analysis of the measurability of the electromagnetic field now can be translated easily and accurately into concrete numbers.

Acknowledgments

The author is grateful to F Frescura for a careful reading of the manuscript. He also acknowledges gratefully correspondence with F Persico, whose searching questions helped the author to realize that the formal expressions for some BR geometric factors imply a suitable regularization.

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